

Möbius Symmetry of Discrete Time Soliton Equations

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Abstract

We have proposed, in our previous papers[1, 2], a method to characterize integrable discrete soliton equations. In this paper we generalize the method further and obtain a q -difference Toda equation, from which we can derive various q -difference soliton equations by reductions.

1 Introduction

It has been known that, for a given soliton equation, there exists a discrete analogue of the evolution equation, which preserves integrability. Once we find a discrete integrable equation, we can derive infinitely many integrable differential equations by taking continuous limits of variables in many different ways. We are interested in characterizing such integrable discrete systems.

In our previous papers [1, 2] we studied various integrable discrete systems which have certain symmetry and satisfy periodic boundary conditions. In this approach the nature of a quadratic equation, imposed by the boundary condition, plays a crucial role. The map which generates a time evolution turns out to be twofold, corresponding to two solutions of the quadratic equation. We showed that the discriminant of the quadratic equation becomes a perfect square when the system is integrable. Owing to this fact the map is free from a square root, hence is rational. We will call this type of map a non square root map (NSRM) in what follows. The discrete versions of Lotka-Volterra equation, KdV equation, Toda lattice, KP equation and Painlevé equations [3, 4, 5, 6, 7, 8, 9, 10] are such examples. The purpose of this paper is to generalize this scheme of characterizing discrete integrable systems.

To clarify the point of our argument let us show first a typical example. The 3-point discrete time Toda lattice is given by the set of coupled equations:

$$\begin{cases} \bar{x}_k \bar{u}_k &= u_k x_{k-1} \\ \bar{x}_k + \bar{u}_{k+1} &= u_k + x_k \end{cases}, \quad k = 1, 2, 3. \quad (1)$$

Here \bar{x}_k means the variable x_k at the time $t + 1$, *i.e.*, $\bar{x}_k(t) = x_k(t + 1)$. We impose the periodic conditions $x_{k+3} = x_k$ and $u_{k+3} = u_k$. Hence there are 6 dependent variables.

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Before solving eq.(1) directly it is more instructive to solve

$$\begin{cases} \bar{x}_k \bar{u}_k &= p_k \\ \bar{x}_k + \bar{u}_{k+1} &= q_k \end{cases} \quad (2)$$

for arbitrary functions p_k, q_k of (x, u) . We can also write them as

$$\begin{cases} \bar{x}_k &= \frac{q_k \bar{x}_{k+1} - p_{k+1}}{\bar{x}_{k+1}} \\ \bar{u}_k &= \frac{q_{k-1} \bar{u}_{k-1} - p_{k-1}}{\bar{u}_{k-1}}. \end{cases} \quad (3)$$

Note that they are special cases of Möbius transformation. Hence by repeating this transformation, say for the first equation of eq.(3), three times, \bar{x}_k is given in terms of $(\bar{x}_{k+3}, \bar{u}_{k-3})$, which are nothing but (\bar{x}_k, \bar{u}_k) themselves owing to the periodic boundary conditions. In this way we obtain an equation

$$\bar{x}_k = \frac{A\bar{x}_k + B}{\Gamma\bar{x}_k + \Delta} \quad (4)$$

or, equivalently,

$$\Gamma\bar{x}_k^2 - (A - \Delta)\bar{x}_k - B = 0, \quad (5)$$

where

$$\begin{aligned} A &= q_1 q_2 q_3 - q_k p_k - q_{k-1} p_{k-1}, \\ B &= -q_k q_{k+1} p_{k+1} + p_{k+1} p_{k-1}, \\ \Gamma &= q_{k+1} q_{k-1} - p_k, \\ \Delta &= -q_{k+1} p_{k+1}, \end{aligned}$$

in the case of three point Toda lattice.

The quadratic equation eq.(5) has two solutions

$$\bar{x}_k = \frac{A - \Delta \pm \sqrt{Dis}}{2\Gamma}, \quad (6)$$

unless its discriminant

$$Dis = (A - \Delta)^2 + 4B\Gamma \quad (7)$$

vanishes. The general form of solutions contain square root of polynomial functions. Under this circumstance a sequence of the map will yield very complicated orbits in general. But the integrable systems are not such cases, *i.e.*, the discriminant happens to be a perfect square of some polynomial. In fact if we substitute the right hand sides of eq.(1) into p_k 's and q_k 's of the expression eq.(6), the discriminant is considerably simplified and is given by

$$Dis = (x_1 x_2 x_3 - u_1 u_2 u_3)^2. \quad (8)$$

This is a typical non square root map (NSRM), which we are going to study (see below for more precise definition).

This remarkable feature could be understood from other point of view as well. Namely by an inspection of our original equations eq.(1) we can see easily that

$$\bar{x}_k = u_k, \quad \bar{u}_k = x_{k-1} \quad (9)$$

is one of sets of solution. This implies that another set of solutions of the quadratic equation eq.(5) must not have a square root either. We would like to emphasize that the existence of the solution of such simple form eq.(9) owes to the symmetry of the equation eq.(1).

We will study, in the following section, N -point maps under which the square roots are removed when periodic boundary conditions are imposed. In section 3 the discrete time Toda equation is studied in detail based on the preceding argument. In particular we will show that this scheme enables one to generalize the Toda equation to a q -difference form, so that various q -difference soliton equations are derived by reductions, including the qP_{IV} equation by Kajiwara, Noumi and Yamada[9]. In section 4, we will discuss time evolution of solutions. The last section is devoted to summarize arguments in this paper.

2 N -point Map with Periodic Boundary Conditions

We consider N -point systems in this section and study conditions under which a map becomes a non square root map (NSRM).

2.1 Möbius transformation

Generalizing eq.(3), let us consider the following sequence of Möbius transformations,

$$\bar{x}_{k+h(n-1)} = \frac{\alpha_n \bar{x}_{k+hn} + \beta_n}{\gamma_n \bar{x}_{k+hn} + \delta_n}, \quad n = 1, 2, \dots, N, \quad h = \pm 1, \quad \alpha_n \delta_n - \beta_n \gamma_n \neq 0 \quad (10)$$

and Möbius matrices

$$M_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}, \quad \det M_n \neq 0, \quad n = 1, 2, \dots, N. \quad (11)$$

After N times of the transformation we obtain the equation

$$\bar{x}_k = \frac{A \bar{x}_{k+hN} + B}{\Gamma \bar{x}_{k+hN} + \Delta}, \quad (12)$$

while the corresponding Möbius map is given by

$$S_N := \begin{pmatrix} A & B \\ \Gamma & \Delta \end{pmatrix} = M_1 M_2 \cdots M_N. \quad (13)$$

We now impose the periodic boundary conditions $\bar{x}_k = \bar{x}_{k+hN}$. Then the formula eq.(12) is equal to the quadratic equation

$$\Gamma \bar{x}_k^2 - (A - \Delta) \bar{x}_k - B = 0. \quad (14)$$

The solutions of this equation are given by

$$\bar{x}_k = \frac{\text{tr } S_N - 2\Delta \pm \sqrt{Dis}}{2\Gamma} \quad (15)$$

whereas the discriminant Dis is

$$Dis = (A - \Delta)^2 + 4B\Gamma = (\text{tr } S_N)^2 - 4 \det S_N. \quad (16)$$

2.2 Non square root maps

In this subsection we study maps such that the discriminant eq.(16) becomes a perfect square of some polynomial function. As we will prove shortly the following 6 patterns of the map M_n satisfy the requirement:

$$\begin{aligned} & \begin{pmatrix} 0 & \beta_n \\ \beta_n & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha_n & 0 \\ 0 & \delta_n \end{pmatrix}, \quad \begin{pmatrix} \alpha_n & \beta_n \\ 0 & \delta_n \end{pmatrix}, \\ & \begin{pmatrix} \alpha_n & 0 \\ \gamma_n & \delta_n \end{pmatrix}, \quad \begin{pmatrix} \gamma_n + \beta_n & -\beta_n \\ \gamma_n & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \beta_n \\ -\gamma_n & \beta_n + \gamma_n \end{pmatrix}. \end{aligned} \quad (17)$$

Proof:

Since the trace and the determinant are invariant under the transposition of the matrices, we do not consider here the fourth and sixth patterns. For the first three patterns we find

$$S_N = \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^N, \quad \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}, \quad \begin{pmatrix} \alpha & F \\ 0 & \delta \end{pmatrix} \quad (18)$$

where

$$\alpha = \alpha_1 \cdots \alpha_N, \quad \beta = \beta_1 \cdots \beta_N, \quad \delta = \delta_1 \cdots \delta_N,$$

and F is a certain function of $\alpha_n, \beta_n, \delta_n$. From this expression we easily find

$$Dis = (2\beta)^2 \delta_{N,odd}, \quad (\alpha - \delta)^2, \quad (\alpha - \delta)^2$$

corresponding to the first, second and the third patterns.

For the 5th pattern we first observe

$$\det S_N = \det M_1 \cdots \det M_N = \beta\gamma. \quad (19)$$

In order to calculate $\text{tr } S_N$, we split M_n into two parts

$$M_n = \begin{pmatrix} \beta_n & -\beta_n \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \gamma_n & 0 \\ \gamma_n & 0 \end{pmatrix} \equiv B_n + C_n. \quad (20)$$

Using the property

$$B_m C_n = 0, \quad m, n \in 1, 2, \cdots N,$$

we can see that S_N has terms of the form $C_1 \cdots C_\rho B_{\rho+1} \cdots B_N$ only. From this fact, the trace of S_N is calculated as

$$\begin{aligned}
\text{tr } S_N &= \sum_{\rho \in N} \text{tr} (C_1 \cdots C_\rho B_{\rho+1} \cdots B_N) \\
&= \sum_{\rho \in N} \text{tr} (C_2 \cdots C_\rho B_{\rho+1} \cdots B_N C_1) \\
&= \text{tr} (B_1 \cdots B_N) + \text{tr} (C_1 \cdots C_N) \\
&= \beta + \gamma
\end{aligned} \tag{21}$$

where $\gamma = \gamma_1 \cdots \gamma_N$. Similarly we can treat the 6th pattern and obtain the same results as eqs.(19) and (22) for the determinant and trace, respectively.

Hence we have obtained discriminants for all six patterns as follows:

$$\begin{aligned}
1) \quad & \text{tr } S_N = \delta_{N, \text{even}} \beta, \quad \det S_N = -\beta^2 \quad \rightarrow \quad Dis = (2\beta)^2 \delta_{N, \text{odd}}, \\
2) \quad & \text{tr } S_N = \alpha + \delta, \quad \det S_N = \alpha \delta \quad \rightarrow \quad Dis = (\alpha - \delta)^2, \\
3) \quad & \text{tr } S_N = \alpha + \delta, \quad \det S_N = \alpha \delta \quad \rightarrow \quad Dis = (\alpha - \delta)^2, \\
4) \quad & \text{tr } S_N = \alpha + \delta, \quad \det S_N = \alpha \delta \quad \rightarrow \quad Dis = (\alpha - \delta)^2, \\
5) \quad & \text{tr } S_N = \gamma + \beta, \quad \det S_N = \beta \gamma \quad \rightarrow \quad Dis = (\gamma - \beta)^2, \\
6) \quad & \text{tr } S_N = \gamma + \beta, \quad \det S_N = \beta \gamma \quad \rightarrow \quad Dis = (\gamma - \beta)^2.
\end{aligned}$$

Therefore every discrete equation with the Möbius matrix pattern of eq.(17) has no square root, although they are not all.

2.3 Rational maps

Solutions of the quadratic equation eq.(14) are given by eq.(15). The purpose of this subsection is to write them explicitly in the case of NSRM. First we note that the 2nd and the 3rd pattern as well as even N case of the 1st pattern do not form a quadratic equation due to $\Gamma = 0$. Hence it is enough to consider the odd N case of 1st pattern, 4th, 5th and 6th patterns.

2.3.1 Odd N 1st pattern

The odd N case is given by the following setup:

$$S_N = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}, \quad \text{tr } S_N = 0, \quad \det S_N = -\beta^2, \quad Dis = (2\beta)^2. \tag{22}$$

Hence the solutions of the quadratic equation are

$$\bar{x}_k = \frac{\text{tr } S_N - 2\Delta \pm \sqrt{Dis}}{2\Gamma} = \frac{\pm 2\beta}{2\beta} = \pm 1. \tag{23}$$

2.3.2 4th pattern

In this pattern we have

$$S_N = \begin{pmatrix} \alpha & 0 \\ F & \delta \end{pmatrix}, \quad \text{tr } S_N = \alpha + \delta, \quad \det S_N = \alpha\delta, \quad Dis = (\alpha - \delta)^2. \quad (24)$$

To calculate F , we split M_n to two parts A_n and D_n as follows:

$$M_n = \begin{pmatrix} \alpha_n & 0 \\ \gamma_n & \delta_n \end{pmatrix} = \begin{pmatrix} \alpha_n & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \gamma_n & \delta_n \end{pmatrix} \equiv A_n + D_n. \quad (25)$$

Using the fact $A_m D_n = 0$, we see that S_N has terms of the form $D_1 \cdots D_{\rho-1} A_\rho \cdots A_N$ only, where $\rho = 1, 2, \dots, N+1$ and $D_0 = A_{N+1} = 0$. All possible terms can be written in such forms as

$$A_1 A_2 \cdots A_N = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad (26)$$

$$D_1 D_2 \cdots D_N = \begin{pmatrix} 0 & 0 \\ \delta_1 \cdots \delta_{N-1} \gamma_N & \delta \end{pmatrix}, \quad (27)$$

$$D_1 \cdots D_{\rho-1} A_\rho \cdots A_N = \begin{pmatrix} 0 & 0 \\ \delta_1 \cdots \delta_{\rho-2} \gamma_{\rho-1} \alpha_\rho \cdots \alpha_N & 0 \end{pmatrix}. \quad (28)$$

Hence we find the expression of F as

$$F = \sum_{\rho=2}^N \delta_1 \cdots \delta_{\rho-2} \gamma_{\rho-1} \alpha_\rho \cdots \alpha_N. \quad (29)$$

The solutions of the quadratic equation are then given as follows:

$$\bar{x}_k = \frac{\text{tr } S_N - 2\Delta \pm \sqrt{Dis}}{2\Gamma} = \frac{(\alpha - \delta) \pm (\alpha - \delta)}{2F} = \begin{cases} \frac{\alpha - \delta}{F} \\ 0 \end{cases}. \quad (30)$$

2.3.3 5th pattern

We already got $\text{tr } S_N$ and $\det S_N$ in the previous subsection, but did not have calculated the elements of S_N . To obtain the solution of the quadratic equation, we must find their elements since information about C and D are necessary for the solutions.

The splitting idea of eq.(20) is convenient to obtain the elements here again. Using this idea, we need to calculate only the following quantities

$$C_1 C_2 \cdots C_N = \gamma \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad (31)$$

$$B_1 B_2 \cdots B_N = \beta \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad (32)$$

$$C_1 \cdots C_{\rho-1} B_\rho \cdots B_N = \gamma_1 \cdots \gamma_{\rho-1} \beta_\rho \cdots \beta_N \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad (33)$$

because $B_m C_n = 0$. Hence Γ and Δ of the elements of S_N are obtained as

$$\Gamma = \gamma_1 \cdots \gamma_N + \sum_{\rho=2}^N \gamma_1 \gamma_2 \cdots \gamma_{\rho-1} \beta_\rho \cdots \beta_N, \quad (34)$$

$$\Delta = - \sum_{\rho=2}^N \gamma_1 \gamma_2 \cdots \gamma_{\rho-1} \beta_\rho \cdots \beta_N. \quad (35)$$

Consequently, noting the fact $\Gamma = \gamma - \Delta$, we obtain the solutions of the quadratic equation as

$$\begin{aligned} \bar{x}_k &= \frac{\text{tr } S_N - 2\Delta \pm \sqrt{Dis}}{2\Gamma} = \frac{(\gamma + \beta) - 2\Delta \pm (\gamma - \beta)}{2(\gamma - \Delta)} \\ &= \begin{cases} 1 \\ \frac{\beta - \Delta}{\gamma - \Delta} \equiv \frac{\beta_N Q_N(\beta, \gamma)}{\gamma_1 Q_1(\beta, \gamma)} \end{cases} \end{aligned} \quad (36)$$

where $Q_n(\beta, \gamma)$ is the homogeneous polynomial of $(\beta_1, \dots, \beta_N, \gamma_1, \dots, \gamma_N)$, that is

$$Q_n(\beta, \gamma) = Q_n(\beta_1, \dots, \beta_N, \gamma_1, \dots, \gamma_N) = \sum_{\rho=2}^N \gamma_{n+1} \cdots \gamma_{n+\rho-1} \beta_{n+\rho} \cdots \beta_{n+N-1}. \quad (37)$$

2.3.4 6th pattern

For this pattern too, we already got $\text{tr } S_N$ and $\det S_N$ in the previous subsection, but did not have calculated the elements of S_N . Using the splitting idea again, the elements of S_N in this pattern are able to be obtained. The splitting of M_n in this case is

$$M_n = \begin{pmatrix} 0 & \beta_n \\ -\gamma_n & \beta_n + \gamma_n \end{pmatrix} = \begin{pmatrix} 0 & \beta_n \\ 0 & \beta_n \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\gamma_n & \gamma_n \end{pmatrix} \equiv B_n + C_n. \quad (38)$$

Since $C_m B_n = 0$, we are enough to calculate the following terms

$$C_1 C_2 \cdots C_N = \gamma \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}, \quad (39)$$

$$B_1 B_2 \cdots B_N = \beta \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad (40)$$

$$B_1 \cdots B_{\rho-1} C_\rho \cdots C_N = \beta_1 \beta_2 \cdots \beta_{\rho-1} \gamma_\rho \cdots \gamma_N \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}. \quad (41)$$

Hence we obtain Γ and Δ of the elements of S_N as

$$\begin{aligned} \Gamma &= -\gamma - \sum_{\rho=2}^N \beta_1 \cdots \beta_{\rho-1} \gamma_\rho \cdots \gamma_N, \\ \Delta &= \beta + \gamma + \sum_{\rho=2}^N \beta_1 \cdots \beta_{\rho-1} \gamma_\rho \cdots \gamma_N. \end{aligned} \quad (42)$$

If we further use $\Gamma = \beta - \Delta$, we get solutions of the quadratic equation as follows:

$$\begin{aligned}\bar{x}_k &= \frac{\text{tr } S_N - 2\Delta \pm \sqrt{Dis}}{2\Gamma} = \frac{(\gamma + \beta) - 2\Delta \pm (\beta - \gamma)}{2(\beta - \Delta)} \\ &= \begin{cases} 1 \\ \frac{\Delta - \gamma}{\Delta - \beta} \equiv \frac{\beta_1 Q_1(\gamma, \beta)}{\gamma_N Q_N(\gamma, \beta)}. \end{cases}\end{aligned}\quad (43)$$

2.4 Polynomial Q

We note that polynomial Q 's, which appear in the solutions of the 5th and 6th patterns, are given by the minor determinants of the diagonal entries of the following matrix.

$$Q(\beta_1, \dots, \beta_N, \gamma_1, \dots, \gamma_N) := \begin{pmatrix} \beta_1 + \gamma_1 & \beta_1 & 0 & 0 & \cdots & 0 & \gamma_1 \\ \gamma_2 & \beta_2 + \gamma_2 & \beta_2 & 0 & \cdots & 0 & 0 \\ 0 & \gamma_3 & \beta_3 + \gamma_3 & & & & \vdots \\ \vdots & & & & & & 0 \\ 0 & 0 & \cdots & & & & \beta_{N-1} \\ \beta_N & 0 & \cdots & 0 & \gamma_N & \beta_N + \gamma_N \end{pmatrix}. \quad (44)$$

For simplicity, we will introduce the following notations.

$$\begin{aligned}Q(\beta, \gamma) &:= Q(\beta_1, \dots, \beta_N, \gamma_1, \dots, \gamma_N), \\ Q(\gamma, \beta) &:= Q(\gamma_1, \dots, \gamma_N, \beta_1, \dots, \beta_N), \\ Q^\vee(\beta, \gamma) &:= Q(\beta_N, \dots, \beta_1, \gamma_N, \dots, \gamma_1), \\ Q^\vee(\gamma, \beta) &:= Q(\gamma_N, \dots, \gamma_1, \beta_N, \dots, \beta_1).\end{aligned}$$

Namely, the matrix Q having entries in ascending order is expressed by Q , and in descending order by Q^\vee . These matrices are related by

$$Q^\vee(\beta, \gamma) = JQ(\gamma, \beta)J, \quad Q(\beta, \gamma) = JQ^\vee(\gamma, \beta)J$$

where

$$J := \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & \cdots & \vdots & \\ 0 & 1 & & \vdots & \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}. \quad (45)$$

We set symbols of them of (n, n) -entry as Q_n , *i.e.*

$$\begin{aligned}
Q_n(\beta, \gamma) &= \sum_{\rho=1}^N \prod_{\mu=1}^{\rho-1} \gamma_{n+\mu} \prod_{\nu=\rho}^{N-1} \beta_{n+\nu}, \\
Q_n(\gamma, \beta) &= \sum_{\rho=1}^N \prod_{\mu=1}^{\rho-1} \beta_{n+\mu} \prod_{\nu=\rho}^{N-1} \gamma_{n+\nu}, \\
Q_n^\vee(\beta, \gamma) &= \sum_{\rho=1}^N \prod_{\mu=1}^{\rho-1} \gamma_{n+N-\mu} \prod_{\nu=\rho}^{N-1} \beta_{n+N-\nu}, \\
Q_n^\vee(\gamma, \beta) &= \sum_{\rho=1}^N \prod_{\mu=1}^{\rho-1} \beta_{n+N-\mu} \prod_{\nu=\rho}^{N-1} \gamma_{n+N-\nu},
\end{aligned}$$

where $\beta_{n\pm N} = \beta_n$, $\gamma_{n\pm N} = \gamma_n$.

These polynomials satisfy the following useful relations.

1. $Q_{N-n+1}(\beta, \gamma) = Q_n^\vee(\gamma, \beta)$, $Q_n(\beta, \gamma) = Q_{N-n+1}^\vee(\gamma, \beta)$,
2. $(\beta_n + \gamma_n)Q_n(\beta, \gamma) = \beta_{n-1}Q_{n-1}(\beta, \gamma) + \gamma_{n+1}Q_{n+1}(\beta, \gamma)$.

The relation 1 is obvious. The relation 2 is non-trivial and will be checked by means of more fundamental relation :

$$\beta_n Q_n(\beta, \gamma) = \beta - \gamma + \gamma_{n+1} Q_{n+1}(\beta, \gamma)$$

where $\beta = \beta_1 \cdots \beta_N$, $\gamma = \gamma_1 \cdots \gamma_N$.

We can show that $Q_n(\beta, \gamma)$ coincides with the cofactor of the (n, n) element of the product RL of the following matrices

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & \beta_N \\ \beta_1 & 1 & & & 0 \\ 0 & \beta_2 & 1 & \cdots & \\ & & & \ddots & \\ 0 & 0 & \cdots & \beta_{N-1} & 1 \end{pmatrix}, \quad R = \begin{pmatrix} \gamma_1 & 1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & 1 & & \\ & & \vdots & & \\ 0 & & & & 1 \\ 1 & 0 & \cdots & 0 & \gamma_N \end{pmatrix}$$

which could represent the dToda equations [3].

2.5 Modification

The discriminant Dis is invariant under arbitrary similarity transformations, since it is given by the trace and determinant of S_N . This fact enables us to modify our map starting from the six patterns we considered above.

Namely under the transformation of M_n ,

$$M_n \rightarrow M_n^{mod} \equiv U_n^{-1} M_n U_{n+1}, \quad (46)$$

the matrix eq.(13) is changed into

$$S_N^{mod} = M_1^{mod} M_2^{mod} \cdots M_N^{mod} = U_1^{-1} M_1 M_2 \cdots M_N U_{N+1}. \quad (47)$$

If we take into account the periodic boundary condition, we must have $U_{N+1} = U_1$. Therefore the trace and the determinant of S_N , hence the discriminant Dis as well, remain constant:

$$\text{tr } S_N^{mod} = \text{tr } S_N, \quad \det S_N^{mod} = \det S_N, \quad Dis^{mod} = Dis, \quad (48)$$

under the transformation eq.(46) for arbitrary matrices U_n 's. This implies that the modified map M_n^{mod} is NSRM if M_n is so.

Now denoting by \bar{x}^{mod} the modified variable of \bar{x} , we will write the Möbius map associated with M_n as $M_n(x_k)$. In this way, the modified Möbius map is possible to be written as

$$\bar{x}_k^{mod} = M_1^{mod}(\bar{x}_{k+1}^{mod}) = U_1^{-1} M_1 U_1(\bar{x}_{k+1}^{mod}). \quad (49)$$

Here if we define $\bar{x}_{k+n}^{mod} = U_{n+1}^{-1}(\bar{x}_{k+n})$, then

$$\begin{aligned} \bar{x}_k^{mod} &= U_1^{-1} M_1 U_2(\bar{x}_{k+1}^{mod}) \\ \Rightarrow U_1^{-1}(\bar{x}_k) &= U_1^{-1} M_1 U_2 U_2^{-1}(\bar{x}_{k+1}) \\ \Rightarrow \bar{x}_k &= M_1(\bar{x}_{k+1}). \end{aligned} \quad (50)$$

As a result, the following two modifications are equivalent

$$\begin{aligned} 1) \quad M_n &\rightarrow M_n^{mod} = U_n^{-1} M_n U_{n+1}, \\ 2) \quad \bar{x}_{k+n} &\rightarrow \bar{x}_{k+n}^{mod} = U_{n+1}^{-1}(\bar{x}_{k+n}). \end{aligned} \quad (51)$$

This correspondence is preserved in the solutions of the quadratic equation, so we can create many equations from the basic six patterns in this way.

3 Discrete Time Toda Equation

We study, in this section, the N -point discrete time Toda equation (dToda) in detail from the view point of the previous section.

3.1 Two types of solution

Let us start with writing the N -point dToda equation.

$$\begin{cases} \bar{x}_k \bar{u}_k &= u_k x_{k-1} \\ \bar{x}_k + \bar{u}_{k+1} &= u_k + x_k \end{cases}, \quad k = 1, 2, 3, \dots, N. \quad (52)$$

In order to apply the argument in the previous section to this system we transform eq.(52) into the form of Möbius transformation,

$$\begin{cases} \bar{x}_k &= \frac{(x_k + u_k)\bar{x}_{k+1} - x_k u_{k+1}}{\bar{x}_{k+1}} \\ \bar{u}_k &= \frac{(x_{k-1} + u_{k-1})\bar{u}_{k-1} - x_{k-2} u_{k-1}}{\bar{u}_{k-1}} \end{cases} \quad (53)$$

corresponding to $(\bar{x}_{k+1}, \bar{u}_{k-1}) \rightarrow (\bar{x}_k, \bar{u}_k)$.

We notice that this is a modified 5th pattern Möbius transformation which is specified by the following data:

$$(\beta_n, \gamma_n) = \begin{pmatrix} (x_{k+n-1}, u_{k+n-1}) \\ (u_{k-n}, x_{k-n}) \end{pmatrix}, \quad U_n = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_n \end{pmatrix}. \quad (54)$$

Substituting these data into eq.(36) we obtain solutions immediately. Before writing them down, however, it is convenient to simplify notations. Namely the matrix Q , which appears in the solutions, is $Q(x_k, x_{k+1}, \dots, x_{k-2}, x_{k-1}, u_k, u_{k+1}, \dots, u_{k-2}, u_{k-1})$. Let the matrix Q be

$$\begin{aligned} Q(x_k, u_k) &:= Q(x_k, \dots, x_{k+N-1}, u_k, \dots, u_{k+N-1}), \\ Q^\vee(x_k, u_k) &:= Q(x_{k+N-1}, \dots, x_k, u_{k+N-1}, \dots, u_k), \end{aligned}$$

and its minor determinants be

$$\begin{aligned} Q_n(x_k, u_k) &:= Q_n(x_k, \dots, x_{k+N-1}, u_k, \dots, u_{k+N-1}), \\ Q_n^\vee(x_k, u_k) &:= Q_n(x_{k+N-1}, \dots, x_k, u_{k+N-1}, \dots, u_k). \end{aligned} \quad (55)$$

These minor determinants satisfy

1. $Q_{N-n+1}(x_k, u_k) = Q_n^\vee(u_k, x_k), \quad Q_n(x_k, u_k) = Q_{N-n+1}^\vee(u_k, x_k),$
2. $(x_{k+n} + u_{k+n})Q_n(x_k, u_k) = x_{k+n-1}Q_{n-1}(x_k, u_k) + u_{k+n+1}Q_{n+1}(x_k, u_k),$
3. $Q_n(x_{k+N}, u_{k+N}) = Q_{n+1}(x_{k+N-1}, u_{k+N-1}) = \dots = Q_{n+N-1}(x_{k+1}, u_{k+1}) = Q_{n+N}(x_k, u_k)$

where $x_{k\pm N} = x_k, u_{k\pm N} = u_k$. Therefore the solutions turn out to be

$$\begin{cases} \begin{cases} \bar{x}_k = x_{k-1} \frac{Q_N(x_k, u_k)}{Q_1(x_k, u_k)} \\ \bar{u}_k = u_k \frac{Q_1(x_k, u_k)}{Q_N(x_k, u_k)} \end{cases} & \text{A-type,} \\ \begin{cases} \bar{x}_k = u_k \\ \bar{u}_k = x_{k-1} \end{cases} & \text{B-type.} \end{cases} \quad (56)$$

Similarly the time reversal map is obtained by writing eq.(52) as

$$\begin{cases} x_k = \frac{(\bar{x}_k + \bar{u}_{k+1})x_{k-1} - \bar{x}_k \bar{u}_k}{x_{k-1}} \\ u_k = \frac{(\bar{x}_k + \bar{u}_{k+1})u_{k+1} - \bar{x}_{k+1} \bar{u}_{k+1}}{u_{k+1}} \end{cases} \quad (57)$$

corresponding to the map $(x_{k-1}, u_{k+1}) \rightarrow (x_k, u_k)$.

This is again a modified 5th pattern of the Möbius transformation. The modification matrix and the elements of the map are given by

$$(\beta_n, \gamma_n) = \begin{pmatrix} (\bar{x}_{k+1-n}, \bar{u}_{k+2-n}) \\ (\bar{u}_{k+n}, \bar{x}_{k-1+n}) \end{pmatrix}, \quad U_n = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_n \end{pmatrix}, \quad (58)$$

and the solutions

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} x_{k-1} = \bar{x}_k \frac{Q_N^\vee(\bar{x}_k, \bar{u}_{k+1})}{Q_1^\vee(\bar{x}_k, \bar{u}_{k+1})} \\ u_k = \bar{u}_k \frac{Q_1^\vee(\bar{x}_k, \bar{u}_{k+1})}{Q_N^\vee(\bar{x}_k, \bar{u}_{k+1})} \end{array} \right. \quad \text{A-type,} \\ \left\{ \begin{array}{l} x_k = \bar{u}_{k+1} \\ u_k = \bar{x}_k \end{array} \right. \quad \text{B-type.} \end{array} \right. \quad (59)$$

Instead of eq.(53) and eq.(57) we could write the dToda equation eq.(52) in the form

$$\left\{ \begin{array}{l} \bar{x}_k = \frac{-x_{k-1}u_k}{\bar{x}_{k-1} - (x_{k-1} + u_{k-1})} \\ \bar{u}_k = \frac{x_{k-1}u_k}{\bar{u}_{k+1} - (x_k + u_k)} \end{array} \right. , \quad (60)$$

$$\left\{ \begin{array}{l} x_k = \frac{-\bar{x}_{k+1}\bar{u}_{k+1}}{x_{k+1} - (\bar{x}_{k+1} + \bar{u}_{k+1})} \\ u_k = \frac{-\bar{x}_k\bar{u}_k}{u_{k-1} - (\bar{x}_{k-1} + \bar{u}_k)} \end{array} \right. \quad (61)$$

corresponding to $(\bar{x}_{k-1}, \bar{u}_{k+1}) \rightarrow (\bar{x}_k, \bar{u}_k)$ and $(x_{k+1}, u_{k-1}) \rightarrow (x_k, u_k)$. These are modified 6th patterns whose elements and the modification matrix are given by

$$(\beta_n, \gamma_n) = \left\{ \begin{array}{l} \left\{ \begin{array}{l} (x_{k-1-n}, u_{k-1-n}) \\ (u_{k+n}, x_{k+n}) \\ (\bar{x}_{k+1+n}, \bar{u}_{k+1+n}) \\ (\bar{u}_{k-n}, \bar{x}_{k-1-n}) \end{array} \right\} \end{array} \right. , \quad U_n = \begin{pmatrix} 1/\gamma_{n-1} & 0 \\ 0 & 1 \end{pmatrix} \quad (62)$$

for eq.(60) and eq.(61), respectively. Applying these data to eq.(43) we obtain the same results of eq.(56) and eq.(59).

3.2 Generalizations

3.2.1 Symmetry of B-type solutions

We have seen that the B-type solution is simply 1 if the map M_n is the form of either 5th or 6th pattern. A modification of the matrix by U_n will change the solutions as well as the equation while the discriminant remains constant. In particular a different B-type solution will correspond to a different set of equations, and hence different A-type solution. If the B-type solution is rational the A-type solution must be also rational satisfying the same equations.

To see how it works we try to generalize the B-type solution eq.(56) to

$$\left\{ \begin{array}{l} \bar{x}_k = u_{k+i} \\ \bar{u}_k = x_{k+j} \end{array} \right. , \quad i, j = 0, 1, 2, \dots, N-1. \quad (63)$$

It is not difficult to find their corresponding equations

$$\left\{ \begin{array}{l} \bar{x}_{k+l}\bar{u}_k = x_{k+j}u_{k+i+l} \\ \bar{x}_k + \bar{u}_{k+m} = x_{k+j+m} + u_{k+i} \end{array} \right. , \quad l, m = 0, 1, 2, \dots, N-1. \quad (64)$$

From the construction this equation has also A-type solution which is a rational polynomial of x and u . And we can set $h = m + l = \pm 1$ without loss of generality.

We would like to emphasize that the B-type solution follows automatically to the symmetry possessed by the equation. In other words, the rational solution of A-type is a consequence of the symmetry of the Toda lattice equation.

3.2.2 q -difference version

In the paper [9] Kajiwara *et al.* have studied affine-Weyl symmetry of the discrete time Painlevé IV equation with parameters, which is called qP_{IV} . We have shown in our previous paper [2] that qP_{IV} is one of the equations which are characterized by the square root free maps.

Now we want to know whether exist similar generalization of the discrete time Toda lattice eq.(52) which we have been studying. To this end we consider the following equation with a set of parameters $a_k, b_k, k = 1, 2, \dots, N$:

$$\begin{cases} \frac{\bar{x}_{k+l} \bar{u}_k}{a_{k+l} b_k} = a_{k+j} x_{k+j} b_{k+i+l} u_{k+i+l} \\ \frac{\bar{x}_k}{a_k} + \frac{u_{k+m}}{b_{k+m}} = a_{k+j+m} x_{k+j+m} + b_{k+i} u_{k+i}. \end{cases} \quad (65)$$

Let us call this equation a q -difference generalized discrete time Toda equation (q Toda). Remarkably this equation has the symmetry of modified 5th or 6th pattern of the Möbius map. So, this is transformed to a quadratic equation of every \bar{x}, \bar{u}, x or u , and has two types of map as solutions to eq.(65).

In fact if we introduce

$$M_n = \begin{pmatrix} a_{k+s+nh} x_{k+s+nh} + b_{k+i+nh} u_{k+i+nh} & -a_{k+s+nh} x_{k+s+nh} \\ b_{k+i+nh} u_{k+i+nh} & 0 \end{pmatrix}, \quad (66)$$

$$U_n = \begin{pmatrix} 1 & 0 \\ 0 & b_{k+i+nh} u_{k+i+nh} \end{pmatrix}, \quad (67)$$

we obtain two maps as follows :

$$\frac{\bar{x}_k}{a_k} = \begin{cases} a_{k+s-h} x_{k+s-h} \frac{Q_N^h(a_{k+s} x_{k+s}, b_{k+i} u_{k+i})}{Q_1^h(a_{k+s} x_{k+s}, b_{k+i} u_{k+i})}, & \text{A-type} \\ b_{k+i} u_{k+i}, & \text{B-type} \end{cases} \quad (68)$$

where $s = j + m, h = l + m$, and Q_n^h is defined by

$$Q_n^h(x_k, u_k) := \begin{cases} Q_n(x_k, u_k) & \text{if } h = 1 \\ Q_n^\vee(x_k, u_k) & \text{if } h = -1. \end{cases}$$

3.2.3 Higher dimension

We can further generalize q Toda to higher dimensions by increasing the number of suffices:

$$(x_k, u_k) \rightarrow (x_{k_1, k_2, \dots, k_M}, u_{k_1, k_2, \dots, k_M}).$$

The B-type map should have the form

$$\begin{cases} \frac{\bar{x}_{k_1, k_2, \dots, k_M}}{a_{k_1, k_2, \dots, k_M}} = b_{k_1+i_1, k_2+i_2, \dots, k_M+i_M} u_{k_1+i_1, k_2+i_2, \dots, k_M+i_M} \\ \frac{\bar{u}_{k_1, k_2, \dots, k_M}}{b_{k_1, k_2, \dots, k_M}} = a_{k_1+j_1, k_2+j_2, \dots, k_M+j_M} x_{k_1+j_1, k_2+j_2, \dots, k_M+j_M}. \end{cases} \quad (69)$$

In the case of $M = 2$, for instance,

$$\begin{cases} \frac{\bar{x}_{k,l}}{a_{k,l}} = b_{k+1, l+1} u_{k+1, l+1} \\ \frac{\bar{u}_{k,l}}{b_{k,l}} = a_{k+1, l-1} x_{k+1, l-1} \end{cases} \quad (70)$$

satisfies

$$\begin{cases} \frac{\bar{x}_{k-1, l} \bar{u}_{k-1, l}}{a_{k-1, l} b_{k-1, l}} = a_{k, l-1} b_{k, l+1} x_{k, l-1} u_{k, l+1}, \\ \frac{\bar{x}_{k, l}}{a_{k, l}} + \frac{\bar{u}_{k-1, l+1}}{b_{k-1, l+1}} = a_{k, l} x_{k, l} + b_{k+1, l+1} u_{k+1, l+1}. \end{cases} \quad (71)$$

This is the q KP equation which we derived in our previous paper.

3.3 Reductions

It has been well known that the discrete time Lotka-Volterra equation (dLV) can be obtained from the discrete time Toda equation by a Miura transformation [5]. We like to show, in this section, that the same dLV-type as well as the discrete time KdV (dKdV)-type equations can be derived from the discrete time Toda lattice eq.(65) by simple reductions. The basic idea of our reduction is to deal with one of two equations in q Toda as a constraint to eliminate half of the variables.

3.3.1 Reduction to the dLV-type

The way of reduction to a dLV-type equation is to impose an additional constraint to the second equation of q Toda as follows.

$$\begin{cases} \frac{\bar{x}_{k+l} \bar{u}_k}{a_{k+l} b_k} = a_{k+j} x_{k+j} b_{k+i+l} u_{k+i+l} \\ \frac{\bar{x}_k}{a_k} + \frac{\bar{u}_{k+m}}{b_{k+m}} = a_{k+j+m} x_{k+j+m} + b_{k+i} u_{k+i} = -r_k \end{cases} \quad (72)$$

where r_k 's are constants.

This constraint enables us to eliminate variables, say u and \bar{u} , from the first equation. Namely the following substitution

$$\begin{cases} \frac{\bar{u}_{k+m}}{b_{k+m}} \rightarrow -\frac{\bar{x}_k}{a_k} - r_k \\ b_{k+i} u_{k+i} \rightarrow -a_{k+j+m} x_{k+j+m} - r_k \end{cases} \quad (73)$$

yields the dLV-type equation

$$\frac{\bar{x}_{k+l}}{a_{k+l}} \left(r_{k-m} + \frac{\bar{x}_{k-m}}{a_{k-m}} \right) = a_{k+j} x_{k+j} (r_{k+l} + a_{k+j+m+l} x_{k+j+m+l}). \quad (74)$$

When $r_k = a_k = b_k = 1$, $k = 1, 2, \dots, N$, this is the dLV equation. Various types of q LV equations which were discussed in our previous paper, are included in this equation.

It is straightforward to obtain two maps of this equation. We find

$$\frac{\bar{x}_k}{a_k} = \begin{cases} a_{k+s-h} x_{k+s-h} \frac{Q_N^h(a_{k+s} x_{k+s}, -r_k - a_{k+j+m} x_{k+j+m})}{Q_1^h(a_{k+s} x_{k+s}, -r_k - a_{k+j+m} x_{k+j+m})}, & \text{A-type} \\ -r_k - a_{k+j+m} x_{k+j+m}, & \text{B-type} \end{cases} \quad (75)$$

which could be also obtained from the map of q Toda eq.(65) by the substitution eq.(73). The qP_{IV} of Kajiwara, Noumi, Yamada [9] is included as one of the A-type map.

3.3.2 Reduction to the dKdV-type

Let us try an alternative case of constraint. Namely we impose a constraint to the first equation of eq.(65).

$$\begin{cases} \frac{\bar{x}_{k+l}}{a_{k+l}} \frac{\bar{u}_k}{b_k} = a_{k+j} x_{k+j} b_{k+i+l} u_{k+i+l} = r_k, \\ \frac{\bar{x}_k}{a_k} + \frac{\bar{u}_{k+m}}{b_{k+m}} = a_{k+j+m} x_{k+j+m} + b_{k+i} u_{k+i}. \end{cases} \quad (76)$$

This enables us to eliminate u and \bar{u} from the second equation according to the rule

$$\begin{cases} \frac{\bar{u}_k}{b_k} \rightarrow \frac{a_{k+l} r_k}{\bar{x}_{k+l}} \\ b_{k+i+l} u_{k+i+l} \rightarrow \frac{r_k}{a_{k+j} x_{k+j}} \end{cases} \quad (77)$$

from which we obtain

$$\frac{\bar{x}_k}{a_k} + \frac{a_{k+l+m} r_{k+m}}{\bar{x}_{k+l+m}} = a_{k+j+m} x_{k+j+m} + \frac{r_{k-l}}{a_{k+j-l} x_{k+j-l}}. \quad (78)$$

This equation contains the discrete time KdV equation as a special case. Hence we call eq.(78) the q KdV equation. The corresponding solutions are given again by simply replacing u and \bar{u} in the solutions of q Toda according to eq.(77).

$$\frac{\bar{x}_k}{a_k} = \begin{cases} a_{k+s-h} x_{k+s-h} \frac{Q_N^h(a_{k+s} x_{k+s}, r_{k-l}/a_{k+j-l} x_{k+j-l})}{Q_1^h(a_{k+s} x_{k+s}, r_{k-l}/a_{k+j-l} x_{k+j-l})}, & \text{A-type} \\ r_{k-l}/a_{k+j-l} x_{k+j-l}. & \text{B-type} \end{cases} \quad (79)$$

3.4 Reflection maps

We define some symbols of the two types of map and their inverse maps of dToda as follows,

$$\begin{aligned} \mathcal{A} &: \begin{pmatrix} x_k \\ u_k \end{pmatrix} \mapsto \begin{pmatrix} x_{k-1} \frac{Q_N(x_k, u_k)}{Q_1(x_k, u_k)} \\ u_k \frac{Q_1(x_k, u_k)}{Q_N(x_k, u_k)} \end{pmatrix}, & \mathcal{B} &: \begin{pmatrix} x_k \\ u_k \end{pmatrix} \mapsto \begin{pmatrix} u_k \\ x_{k-1} \end{pmatrix}, \\ \mathcal{A}^{-1} &: \begin{pmatrix} x_{k-1} \\ u_k \end{pmatrix} \mapsto \begin{pmatrix} x_k \frac{Q_N^\vee(x_k, u_{k+1})}{Q_1^\vee(x_k, u_{k+1})} \\ u_k \frac{Q_1^\vee(x_k, u_{k+1})}{Q_N^\vee(x_k, u_{k+1})} \end{pmatrix}, & \mathcal{B}^{-1} &: \begin{pmatrix} x_k \\ u_k \end{pmatrix} \mapsto \begin{pmatrix} u_{k+1} \\ x_k \end{pmatrix}. \end{aligned}$$

These maps satisfy the following relations.

$$\begin{aligned} \text{(I)} \quad & \mathcal{A}\mathcal{A}^{-1} = \mathcal{A}^{-1}\mathcal{A} = 1, \quad \mathcal{B}\mathcal{B}^{-1} = \mathcal{B}^{-1}\mathcal{B} = 1, \\ \text{(II)} \quad & \mathcal{A}\mathcal{B}^2 = \mathcal{B}^2\mathcal{A}, \\ \text{(III)} \quad & \mathcal{A}\mathcal{B}\mathcal{A} = \mathcal{B}^3, \end{aligned} \tag{80}$$

or

$$\begin{aligned} \text{(I')} \quad & \mathcal{B}\mathcal{B}^{-1} = \mathcal{B}^{-1}\mathcal{B} = 1, \\ \text{(II')} \quad & \mathcal{C}\mathcal{B} = \mathcal{B}\mathcal{D}, \\ \text{(III')} \quad & \mathcal{C}^2 = \mathcal{D}^2 = 1, \end{aligned} \tag{81}$$

where $\mathcal{C} = \mathcal{A}^{-1}\mathcal{B} = \mathcal{B}^{-1}\mathcal{A}$ and $\mathcal{D} = \mathcal{A}\mathcal{B}^{-1} = \mathcal{B}\mathcal{A}^{-1}$. The relations eq.(81) are derived from eq.(80) by some calculations, automatically.

The reflection maps \mathcal{C} , \mathcal{D} change the inverse A-type map to the inverse B-type one and the A-type one to the B-type one, respectively. Namely,

$$\begin{aligned} \mathcal{C} &: \text{inverse A-type} \mapsto \text{inverse B-type}, \\ & \quad \text{inverse B-type} \mapsto \text{inverse A-type}, \\ \mathcal{D} &: \text{A-type} \mapsto \text{B-type}, \\ & \quad \text{B-type} \mapsto \text{A-type}. \end{aligned}$$

Hence, these reflection maps are permutations of solutions of the quadratic equation eq.(14) in the case of dToda, *i.e.* Galois transformations of eq.(14). Furthermore, the generalized version of dToda eq.(64) satisfy these relations eq.(80) or eq.(81), but q Toda does not. These reflection maps play an important role in affine-Wye symmetries and q -difference Painlevé equations as studied by Kajiwara, *et al.* [9, 10].

4 Discrete Time Evolution of the q Toda

A sequence of B-type map does not generate an orbit but causes exchange of the variables x and u , whereas an A-type map generates an orbit. We will study a discrete time evolution of A-type map in this section. To make the time dependence explicit we write the variables \bar{x} and \bar{u} as x^{t+1} and u^{t+1} in the following.

4.1 Time evolution of the A-type maps

A repeated use of eq.(56) enables one to express x_k^{t+1} and u_k^{t+1} in terms of x_k^0 's and u_k^0 's as

$$\begin{cases} x_k^{t+1} = x_{k-1}^t \frac{Q_{k-1}^t}{Q_k^t} = \cdots = x_{k-t-1}^0 \prod_{s=0}^t \frac{Q_{k-s-1}^{t-s}}{Q_{k-s}^{t-s}}, \\ u_k^{t+1} = u_k^t \frac{Q_k^t}{Q_{k-1}^t} = \cdots = u_k^0 \prod_{s=0}^t \frac{Q_k^{t-s}}{Q_{k-1}^{t-s}}. \end{cases} \quad (82)$$

Here we denote by Q_k^t the following polynomial of x 's and u 's,

$$Q_k^t := \sum_{j=1}^N u_{k+1}^t u_{k+2}^t \cdots u_{k+j-1}^t x_{k+j}^t \cdots x_{k+N-1}^t. \quad (83)$$

If we substitute eq.(82) into the right hand side of eq.(83) we will find a considerable cancellation of factors in the numerator and the denominator. The cancellation might be associated with a reduction of algebraic entropy [11, 12]. As a consequence we obtain a recurrence equation for Q^t :

$$Q_k^{t+1} = \sum_{j=1}^N u_{k+1}^0 u_{k+2}^0 \cdots u_{k+j-1}^0 x_{k+j-t-1}^0 \cdots x_{k+N-t-2}^0 \prod_{s=0}^t \frac{Q_{k+j-1}^{t-s} Q_{k+j-s-1}^{t-s}}{Q_k^{t-s} Q_{k+N-s-1}^{t-s}}. \quad (84)$$

The A-type solutions are given if Q^t is found by solving eq.(84) iteratively. The generalization of this result to the case of q Toda is straightforward. This is, however, a cumbersome task. And so, we will look orbits of A-type map described by numerical computation.

4.2 Numerical observation of ultra-discrete version of the A-type maps

It is not difficult to see behaviour of orbits of the A-type map eq.(82) as long as we generate them numerically by using computers. They are constrained on curves since there are sufficient number of conserved quantities. If parameters a_k 's and b_k 's are introduced the orbits tend to expand and closed curves turn to spirals. These are behaviours which we could expect naively from our previous experiences. Instead of studying them in detail we would like to present here behaviour of ultra-discrete version of solutions.

It has been known that if a discrete integrable equation has its ultra-discrete version the integrability is preserved [13]. There exists a systematic method to ultra-discretize a discrete integrable equation if it contains only product and summation of variables but not subtractions. Namely we introduce a new variable X and replace x by $x = e^{X/\epsilon}$. The ultra-discretization procedure [13] is defined by

$$\lim_{\epsilon \rightarrow +0} (e^{X/\epsilon} \cdot e^{Y/\epsilon}) \rightarrow X + Y, \quad \lim_{\epsilon \rightarrow +0} (e^{X/\epsilon} + e^{Y/\epsilon}) \rightarrow \max(X, Y).$$

To be specific let us consider the case of dKdV-type equation. The q -difference version has been given by eq.(78). If we further specify to 3-point case the equations are

$$\frac{\bar{x}_k}{a_k} + \frac{a_{k+1}}{\bar{x}_{k+1}} = a_{k+1+j} x_{k+1+j} + \frac{1}{a_{k+j} x_{k+j}}, \quad (85)$$

Figure 1: $A_1 = A_2 = A_3 = 0$

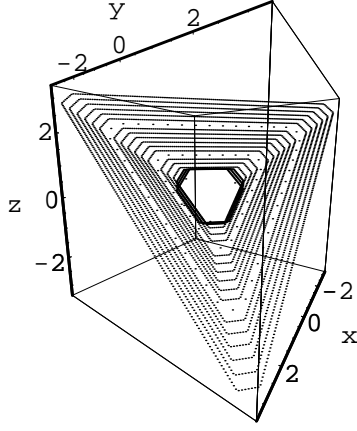
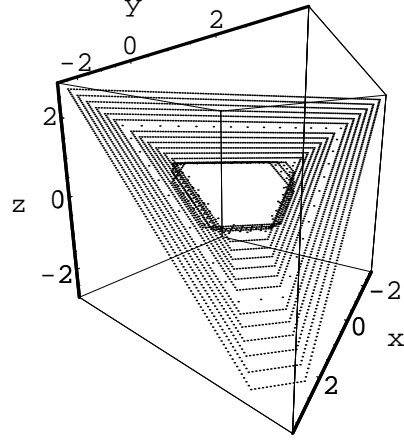


Figure 2: $j = 0, A_1 + A_2 + A_3 = 0$



$$k = 1, 2, 3, \quad j = 0, 1, 2, \quad a_{k+3} = a_k, \quad x_{k+3} = x_k.$$

The corresponding A-type map can be also found readily from the one of q -Toda equation by the reduction eq.(77).

$$\bar{x}_{k-j} = a_{k-j}a_{k+2}x_{k+2} \frac{1 + a_{k+2}a_{k+1}x_{k+2}x_{k+1} + a_k a_{k+1}^2 a_{k+2} x_k x_{k+1}^2 x_{k+2}}{1 + a_k a_{k+2} x_k x_{k+2} + a_k a_{k+1} a_{k+2}^2 x_k x_{k+1} x_{k+2}^2}, \quad (86)$$

$$k = 1, 2, 3, \quad j = 0, 1, 2, \quad a_{k+3} = a_k, \quad x_{k+3} = x_k.$$

The ultra-discrete version of eq.(86) turns to

$$\begin{aligned} \bar{X}_{k-j} = & A_{k-j} + A_{k+2} + X_{k+2} \\ & + \max(0, A_{k+2} + A_{k+1} + X_{k+2} + X_{k+1}, A_k + 2A_{k+1} + A_{k+2} + X_k + 2X_{k+1} + X_{k+2}) \\ & - \max(0, A_k + A_{k+2} + X_k + X_{k+2}, A_k + A_{k+1} + 2A_{k+2} + X_k + X_{k+1} + 2X_{k+2}). \end{aligned} \quad (87)$$

Here we set $a_k = e^{A_k/\epsilon}$, $k = 1, 2, 3$. Let X_k 's and A_k 's be rational numbers and impose a condition $A_1 + A_2 + A_3 = 0$ so that the map returns to initial values after some steps. Under these circumstances we find maps such as presented in Figures 1 to 4.

Figure 1 shows the case of $A_k = 0$, $k = 1, 2, 3$ for all $j=0,1,2$. Similarly Fig. 2 corresponds to the case of $A_1 + A_2 + A_3 = 0$ and $j = 0$. The straight lines in these figures are known to correspond to integrable orbits [14]. Figures 3 and 4 are the cases of $j = 1$ and $j = 2$ respectively. They do not seem to describe integrable orbits. Therefore the orbits behave differently for different value of j . This must be contrasted with the case of dKdV in which orbits form smooth closed curves under the condition $a_1 a_2 a_3 = 1$ for all $j = 0, 1, 2$.

It will be worth while to point out that the dKdV map eq.(86) in the case of $j = 0$ is

Figure 3: $j = 1, A_1 + A_2 + A_3 = 0$

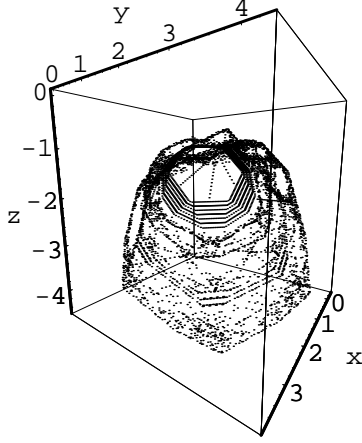
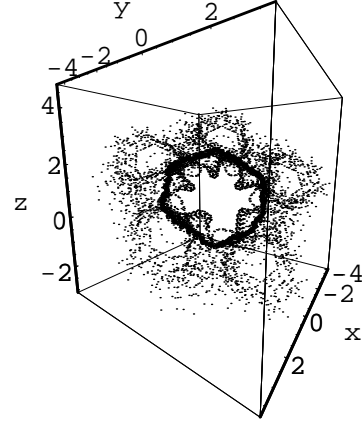


Figure 4: $j = 2, A_1 + A_2 + A_3 = 0$



invariant under the following Bäcklund transformation

$$\begin{aligned} \pi(a_k) &= a_{k+1}, & \pi(x_k) &= x_{k+1}, \\ s_i(a_j) &= a_j(a_{i+1}a_{i-1})^{\lambda_{ij}}, & s_i(x_j) &= x_j \left(\frac{a_{i+1}a_{i-1} + x_{i+1}x_{i-1}}{1 + a_{i+1}a_{i-1}x_{i+1}x_{i-1}} \right)^{\gamma_{ij}}, \end{aligned}$$

where

$$\Lambda := (\lambda_{ij}) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad \Gamma := (\gamma_{ij}) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

$$s_k^2 = 1, \quad \pi^3 = 1, \quad s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}, \quad \pi s_k = s_{k+1} \pi$$

and $i, j, k = 1, 2, 3$ and $a_{k+3} = a_k, x_{k+3} = x_k$. This is the same affine Weyl symmetry shared by qP_{IV} [9]. But $N(\geq 4)$ -point dKdV maps of $j = 0$ does not have this symmetry even though it exhibits an integrable orbit.

We have also studied 3-point ultra-discrete LV-type maps in the same manner. The results are similar to those of KdV-type maps and found one integrable map corresponding to the qP_{IV} .

5 Summary

The main purpose of this paper was to clarify the method characterizing discrete integrable systems, which we introduced in our previous works. We have shown that, in the case of integrable systems, two adjacent variables are related by a Möbius transformation of special forms. We found at least 6 patterns of such forms.

This observation enabled us to obtain quite large class of new discrete integrable systems. The generalized discrete Toda equations (dToda) of eq.(64) is one of such examples.

By certain reductions of the dToda equation dKdV and dLV were obtained. The method admits introduction of arbitrary parameters into the equations and further generalization to q Toda systems.

If we solve the equations in the form in which each variable is given as a function of variables in the previous time, there exist always two types of solutions which we called A-type and B-type. The B-type map does not create orbit but exchanges variables, while a sequence of the A-type map generates an orbit.

The map of A-type is written as a rational polynomial of variables whose numerator and denominator are expressed by minor determinants of the matrix Q in eq.(44). It is interesting to notice that this matrix Q is the same one known in the field of affine Weyl group and q -Painlevé equations [9, 10].

We have not succeeded analyzing full behaviour of a sequence of A-type maps. It is one of the most interesting problems to find behaviour of the map when parameters a_k 's and b_k 's are introduced. Some numerical results using ultra-discrete method have shown that the introduction of the parameters preserve integrability in some cases but not always. This result seems to show that the non square root map condition is not sufficient but necessary for a discrete map being integrable. It is desirable to clarify this point further.

References

- [1] Y. Narita, S. Saito, N. Saitoh and K. Yoshida : J. Phys. Soc. Jpn. **70** (2001) 1246.
- [2] S. Saito, N. Saitoh, J. Yamamoto and K. Yoshida : J. Phys. Soc. Jpn. **70** (2001) 3517.
- [3] R. Hirota, S. Tsujimoto and T. Imai : “*Difference Scheme of Soliton Equations*”, in *Future Directions of Nonlinear Dynamics in Physical and Biological Systems*, ed. by P.L.Christiansen et al., (Plenum Press, New York, 1993) p.7.
- [4] Y. Ohta, R. Hirota and S. Tsujimoto : J.Phys. Soc. Jpn. **62** (1993) 1872.
- [5] R. Hirota, and S. Tsujimoto : J.Phys.Soc.Jpn. **64** (1995) 3125-3127
- [6] M. Jimbo and T. Miwa : J. Phys. Soc. Jpn. **51** (1982) 4116, 4125.
- [7] B. Kupershmidt : *KP or mKP Mathematical Surveys and Monographs Vol 78* (American Mathematical Society, 2000) and references therein.
- [8] Y. Suris : see, for instance, arXiv:solv-int/9902003.
- [9] K. Kajiwara, M. Noumi and Y. Yamada : arXiv:nlin.SI/0012063,
Kenji Kajiwara, Masatoshi Noumi and Yasuhiko Yamada : arXiv:nlin.SI/0106029,
Kenji Kajiwara, Masatoshi Noumi and Yasuhiko Yamada : arXiv:nlin.SI/0112045.
- [10] T. Masuda : arXiv:nlin.SI/0107050,
T. Masuda : arXiv:nlin.SI/0202044.

- [11] J. Hietarinta and C.Viallet : Phys. Rev. Lett. **81** (1998) 325,
- [12] S. Lafortune, A.Ramani, B.Grammaticos, Y.Ohta and K.M.Tamizhmani :
arXiv:nlin.SI/0104020
- [13] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma : Phys. Rev. Lett. **76**
(1996) 3247-3250
- [14] R. Hirota, M. Iwao, A. Ramani, D. Takahashi, B. Grammaticos and Y. Ohta : Phys.
Lett. A **236** (1997) 39–44